Localized Nonlinear Waves in Nonlinear Schrödinger Equation with Nonlinearities Modulated in Space and Time

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In this paper, the generalized sub-equation method is extended to investigate localized nonlinear waves of the one-dimensional nonlinear Schrödinger equation (NLSE) with potentials and nonlinearities depending on time and on spatial coordinates. With the help of symbolic computation, three families of analytical solutions of this NLS-type equation are presented. Based on these solutions, periodically and quasiperiodically oscillating solitons (dark and bright) and moving solitons are observed. Some implications to Bose–Einstein condensates are also discussed.

Key words: Nonlinear Schrödinger Equation; Time- and Space-Modulated Nonlinearities; Solitons; Bose–Einstein Condensates.

1. Introduction

The nonlinear Schrödinger equation (NLSE) is one of the most important and universal nonlinear models of modern science. It appears in many branches of physics and applied mathematics, including nonlinear optics [1], Bose–Einstein condensates (BECs) [2 – 5], biomolecular dynamics [6], and so on. Especially, with the development of optical soliton communication and the experimental realization of BECs, there have been many theoretical and experimental investigations in models based on the NLSE during the last few years.

Various nonlinear excitations in BECs such as dark and bright solitons [7 – 11], vortices [12, 13], BEC dynamics in optical lattices [14, 15], and two-component BECs [16] have been observed and studied. Theoretical and experimental studies have shown that the properties of BECs, including their shape and collective nonlinear excitations, are determined by the sign and magnitude of the s-wave scattering length, which can be controlled by means of the external magnetic or low-loss optical Feshbach-resonance (FR) technique [17 – 19]. These techniques offer us some opportunities to get a spatiotemporal management of the local nonlinearity through the use of time-dependent and/or nonuniform fields.

In nonlinear optics, after predictions of the possibility of the existence [20] and experimental discovery by Mollenauer et al. [21], today, NLSE optical solitons are regarded as the natural data bits and as an important alternative for the next generation of ultrahigh speed optical telecommunication systems [1, 22 – 27]. Recent developments [28 – 30] have led to the discovery of new classes of waves, such as the so-called optical similariton and nonautonomous solitons, which arise when the interaction of nonlinearity, dispersion, and gain in a high-power fiber amplifier causes the shape of an arbitrary input pulse to converge asymptotically to a pulse whose shape is selfsimilar.

Since it is believed that atomic matter nonlinear excitations are of importance for the development of applications of BECs, it is of interest to develop some new mathematical algorithms or extend some known effective methods to investigate some exact solutions, especially bright and dark solitons, in realistic models. With this motivation, in this work we will extend the generalized subequation method [31] to explore some exact solutions of the physical systems ruled by the NLSE of the general form [32]

$$\mathrm{i}\,\psi_t = -\psi_{xx} + v(x,t)\,\psi + g(x,t)|\psi|^2\psi. \qquad (1)$$

In the case of BECs, $\psi = \psi(x,t)$ represents the macroscopic wavefunction, $v(x,t)$ is a space-dependent external potential which oscillates periodically in time from attractive to expulsive behaviour, and $g(x,t)$ describes the modulation of the nonlinearity in space and time. The signs of $g(t,x)$ can be positive or negative, indicating that the interactions are repulsive or attractive, respectively. In [32], the authors constructed explicit nontrivial solutions of (1) by using a similarity transformation and gave some implications of the field of matter waves. When $v(x,t) = v(x)$ and $g(x,t) = g(x)$, Belmonte-Beitia et al. constructed explicit solutions for (1) by Lie group theory and canonical transformation and discussed their applications to the field of nonlinear matter waves [33]. When the atomic scattering length $g(x,t)$ is only time-dependent, and $v(x,t)$ takes various different potentials, such as a parabolic potential or a combined potential, many authors investigated (1) from different view points by different methods [23, 24, 28–31, 34–40].

The paper is organized as follows: In Section 2, we extend the generalized sub-equation method [31] to (1) and successfully construct three families of analytical solutions of it. In Section 3, we give the expressions of periodically and quasiperiodically oscillating solitons (dark and bright) and moving solitons. Finally, some conclusions are given briefly.

2. Exact Solutions of NLSE Systems with Time- and Space-Modulated Nonlinearities

We now extend the generalized sub-equation method [31] to investigate some exact solutions for (1). According to the idea of the method, balancing the highest-order derivative term and the nonlinear terms, we assume the solutions of (1) as of the following general form:

$$\psi = \left[\frac{A_0(x,t) + A_1(x,t)\phi(\xi) + B_1(x,t)\phi'(\xi)}{1 + a_1(x,t)\phi(\xi) + b_1(x,t)\phi'(\xi)}\right] \cdot \exp[\mathrm{i}\Theta(x,t)], \tag{2}$$

where $A_0(x,t)$, $A_1(x,t)$, $B_1(x,t)$, $a_1(x,t)$, $b_1(x,t)$, $\theta(x,t)$ are undetermined functions and $\phi(\xi)$ is determined by

$$\phi'^2(\xi) = h_0 + h_1\phi(\xi) + h_2\phi^2(\xi) + h_3\phi^3(\xi) + h_4\phi^4(\xi) \tag{3}$$

with $\xi \equiv \xi(x,t)$ and h_i $(i = 0,1,2,3,4)$ being arbitrary constants, where the prime denotes differentiation with respect to ξ.

Next, substituting (2) with (3) into (1) at same time, we take a new function

$$\xi = F(X),\ X \equiv X(x,t) = \gamma(t)x + \sigma(t), \tag{4}$$

where $\gamma(t)$ is the inverse of the width of the localized solution and $-\sigma(t)/\gamma(t)$ is the position of its center of mass. Then we get a set of huge numbers of differential equations (for simplification, we omit the set in this paper). After some thorough analysis and some quite tedious calculations, three families of exact solutions for (1) are obtained under the following constraint conditions:

$$\begin{aligned} g(x,t) &= M\gamma(F')^3, \\ v(x,t) &= w(t)x^2 + f(t)x + h(x,t), \\ \Theta(x,t) &= -\frac{\gamma_t}{4\gamma}x^2 - \frac{\sigma_t}{2\gamma}x + \rho, \end{aligned} \tag{5}$$

where

$$\begin{aligned} w(t) &= \frac{\gamma_{tt}}{4\gamma} - \frac{\gamma_t^2}{2\gamma},\ f(t) = \frac{\sigma_{tt}}{2\gamma} - \frac{\sigma_t\gamma_t}{\gamma^2}, \\ h(x,t) &= \frac{[3(F''')^2 - 2F'''F]\gamma^2}{4(F')^2} + N\gamma^2(F')^2 \\ &\quad - \frac{\sigma_t^2}{4\gamma^2} - \rho_t \end{aligned} \tag{6}$$

with ρ as arbitrary function of t and M, N as constants which are satisfied with different conditions in different solutions.

Family 1. When $h_1 = h_3 = 0$, the following series of solutions of (1) can be derived:

$$\psi_{\mathrm{I}} = C_1\sqrt{\frac{\gamma}{F'}}\phi(\xi)\exp[\mathrm{i}\Theta(x,t)], \tag{7}$$

$$M = \frac{2h_4}{C_1^2}, N = h_2, \tag{8}$$

where $\Theta(x,t)$ and γ, F are determined by (4)–(6), C_1, h_2, h_4 are arbitrary constants, and $\phi(\xi)$ can be taken as one of 33 solutions arranged in Table 1 of [41], which include hyperbolic function solutions, Jacobi elliptic function solutions, trigonometric function solutions, etc. For simplification, we do not list them in this paper.

Family 2. When $h_2 = h_4 = 0$, the following two types of solutions of (1) can be obtained:
Case 2.1

$$\psi_{\mathrm{II}}^1 = \frac{4h_0C_2\phi(\xi)}{[4h_0 + h_1\phi(\xi)]}\sqrt{\frac{\gamma}{F'}}\exp[\mathrm{i}\Theta(x,t)], \tag{9}$$

$$M = \frac{5h_1^4}{128h_0^3C_2^2},\ N = -\frac{3h_1^2}{8h_0},\ h_3 = -\frac{h_1^3}{8h_0^2}. \tag{10}$$

Case 2.2

$$\psi_{\mathrm{II}}^2 = \frac{3C_3[h_0 - 2h_1\phi(\xi)]}{[3h_0 + 2h_1\phi(\xi)]}\sqrt{\frac{\gamma}{F'}}\exp[\mathrm{i}\Theta(x,t)], \tag{11}$$

$$M = -\frac{7h_1^2}{9C_3^2 h_0},\; N = \frac{3h_1^2}{h_0},\; h_3 = \frac{4h_1^3}{h_0^2}, \tag{12}$$

where $\Theta(x,t)$, γ, and F are determined by (4)–(6), C_2, C_3, h_0, h_1 are all non-zero constants, and $\phi(\xi)$ is the following Weierstrass elliptic doubly periodic solution:

$$\phi(\xi) = \wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right),\; h_3 > 0, \quad g_2 = -4\frac{h_1}{h_3},\; g_3 = -4\frac{h_0}{h_3}. \tag{13}$$

Family 3. When $h_0 = h_1 = 0$, three kinds of solutions of (1) can be derived as follows:

Case 3.1

$$\psi_{\mathrm{III}}^1 = \pm\frac{\phi(\xi)}{[1+\mu\phi(\xi)]}\sqrt{\frac{(2h_4 - \mu h_3)\gamma}{MF'}} \cdot \exp[\mathrm{i}\Theta(x,t)], \tag{14}$$

$$N = \frac{h_3}{2\mu},\; h_2 = \frac{h_3}{2\mu}. \tag{15}$$

Case 3.2

$$\psi_{\mathrm{III}}^2 = \pm\frac{[h_3 - (\mu h_3 - 4h_4)\phi(\xi)]}{4[1+\mu\phi(\xi)]}\sqrt{\frac{2\gamma}{h_4 MF'}} \cdot \exp[\mathrm{i}\Theta(x,t)], \tag{16}$$

$$N = -\frac{h_3^2}{8h_4},\; h_2 = \frac{h_3^2}{4h_4}. \tag{17}$$

Case 3.3

$$\psi_{\mathrm{III}}^3 = \pm\frac{4h_4\phi'(\xi)}{[h_3 + 4h_4\phi(\xi)]}\sqrt{\frac{2\gamma}{MF'}} \cdot \exp[\mathrm{i}\Theta(x,t)], \tag{18}$$

$$N = \frac{h_3^2}{4h_4},\; h_2 = \frac{h_3^2}{4h_4}, \tag{19}$$

where $\Theta(x,t)$ and γ, F are determined by (4)–(6), μ, h_3, h_4 are all non-zero constants, and $\phi(\xi)$ is one of the following two hyperbolic function solutions:

$$\phi_{\mathrm{III}}^1(\xi) = \frac{4h_2H_0\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{2\Omega_1 + 2(1+\Delta_1)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - (2h_3H_0 + \Omega_1)\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}, \tag{20}$$

$$\phi_{\mathrm{III}}^2(\xi) = \frac{4h_2H_0\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{2\Omega_2 - 2(\Delta_2 + H_0^2)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - (2h_3H_0 + \Omega_2)\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}, \tag{21}$$

where $h_2 > 0$, $H_0 = \exp(\sqrt{h_2}H_1)$ is an arbitrary constant, and $\Delta_1 = H_0{}^2(4h_2h_4 - h_3{}^2)$, $\Delta_2 = (4h_2h_4 - h_3{}^2)$, $\Omega_1 = 1 - \Delta_1$, and $\Omega_2 = H_0{}^2 - \Delta_2$.

Thus, by selecting $\gamma(t)$, $\sigma(t)$, $\alpha(t)$, and $F(X)$, we can generate pairs $v(x,t)$, $g(x,t)$, and obtain corresponding analytical solutions from the above solutions. It is necessary to point out that the results in [32] can be reproduced from our Family 1 by setting $h_0 = 0$, $h_2 = \mu$, $h_4 = \frac{G}{2}$, and $C_1 = 1$. But to our knowledge, the other solutions obtained here have not been reported earlier.

3. Periodically and Quasiperiodically Oscillating Soliton and Moving Soliton

In order to understand the significance of these solutions in Families 1–3 obtained in Section 2, we are more interested in the main soliton features of them. In the following, we mainly consider two examples of these derived solutions as application.

Now, we are more willing to focus attention to the case of specific nonlinearity which may produce some example of interest. In fact, the choice of possible nonlinearity is very rich, here we suppose that it is given explicitly by

$$g(x,t) = M\gamma(t)[1+\lambda\exp(-X^2)]^3, \tag{22}$$

where λ is a real parameter which controls the behaviour of the nonlinearity. This nonlinearity can be obtained by the application of three modulated Gaussian laser beams on the BEC, as experimentally demonstrated [34] to realize optically controlled interactions via the optical Feshbach resonance. To better see this, we expand the term in (22) to get

$1+3\lambda\,\mathrm{e}^{-X^2}+3\lambda^2\,\mathrm{e}^{-2X^2}+\lambda^3\,\mathrm{e}^{-3X^2}$, with each Gaussian term representing the action of a laser beam with properly adjusted intensity, frequency, and waist [42]. At the same time, we may choose the function $F(X)$ as

$$F(X)=\frac{1}{2}\lambda\sqrt{\pi}\mathrm{erf}(X)+X. \quad (23)$$

To make sure that frequency $w(t)$ and nonlinearity $g(x,t)$ are bounded for realistic case, the inverse of the width of the localized excitation $\gamma(t)$ is assumed as the complex periodic function

$$\gamma(t)=0.1+[\alpha_0+\alpha\mathrm{dn}(k_1t,n_1)+\beta\mathrm{dn}(k_2t,n_2)]^2, \quad (24)$$

where α_0, α, β, k_1,k_2 are real constants and n_1, $n_2\in[0,1]$ are the modules of the Jacobi elliptic functions.

3.1. Periodically and Quasiperiodically Oscillating Soliton

Firstly, we consider the particular case when $\sigma(t)=0$.

i. One example is (7) and (8) with $\phi(\xi)$ expressed by the Jacobi elliptic function $\mathrm{cn}(\xi;m)$ in [41] which has the form

$$\psi_{\mathrm{I}}^1=C_1\sqrt{\frac{-h_2m^2\gamma}{h4(2m^2-1)F'}}\mathrm{cn}\left(\sqrt{\frac{h_2}{2m^2-1}}\xi;m\right)\cdot\exp[\mathrm{i}\Theta(x,t)], \quad (25)$$

where $h_2>0$, $h_4<0$, and $h_0=\frac{h_2^2m^2(1-m^2)}{h_4(2m^2-1)^2}$.

The corresponding nonlinearity and the potential in (4)–(6) has to be given by the form

$$g(x,t)=\frac{2h_4\gamma(1+\lambda\,\mathrm{e}^{-\gamma^2x^2})^3}{C_1^2}, \quad (26)$$

$$v(x,t)=w(t)x^2-\frac{\lambda\gamma^4(2\mathrm{e}^{\gamma^2x^2}-\lambda)x^2}{(\mathrm{e}^{\gamma^2x^2}+\lambda)^2}+\frac{h_2\gamma^2(\mathrm{e}^{\gamma^2x^2}+\lambda)^2}{\mathrm{e}^{2\gamma^2x^2}}+\frac{\lambda\gamma^2}{\mathrm{e}^{\gamma^2x^2}+\lambda}-\rho_t. \quad (27)$$

In Figure 1, we plot $w(t)$ in two cases to illustrate its periodic and quasiperiodic features. In Figure 2, we note that the potentials periodically oscillate from attractive to expulsive behaviour mainly, except for the small attractive or expulsive structures near the origin. However, when this small allowed difference exist, the

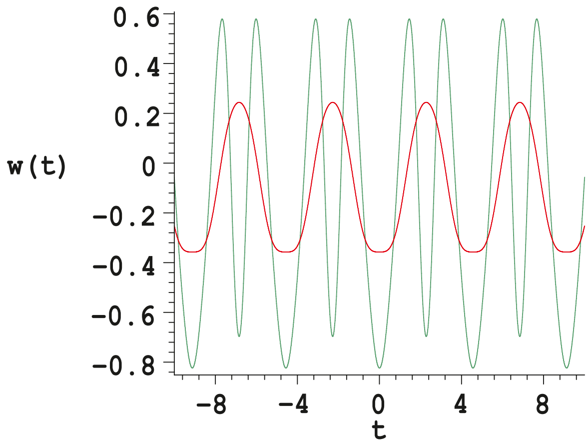

Fig. 1 (colour online). Plots of $w(t)$ in (6), for $\alpha_0=0.01$, $\alpha=0.9$, $\beta=0$, $k_1=1$, $n_1=0.9$, and for $\alpha_0=0.01$, $\alpha=1$, $\beta=0.6$, $k_1=1$, $k_2=2$, $n_1=n_2=0.9$, respectively.

solutions maintain the qualitative behaviour which is displayed in Figure 3. So we deduce that (25) evolve in time periodically or quasiperiodically, depending on the way $\gamma(t)$ in (24), showing that they are localized excitations which we name periodic and quasiperiodic bright solitons.

It is easy to get the dark soliton solutions of (1) if we choose $\phi(\xi)$ expressed by the Jacobi elliptic function $\mathrm{sn}(\xi;m)$ in [41] when $m\to1$ in (7) and (8). Here, we do not list this situation in this paper.

ii. Another example is (9) and (10); we rewrite it here by the form

$$\psi_{\mathrm{II}}^1=\frac{4h_0C_2\sqrt{\gamma}\wp\left(\frac{\sqrt{h_3}}{2}\xi,g_2,g_3\right)}{\sqrt{F'}\left[4h_0+h_1\wp\left(\frac{\sqrt{h_3}}{2}\xi,g_2,g_3\right)\right]}\cdot\exp[\mathrm{i}\Theta(x,t)], \quad (28)$$

where $h_3>0$, $g_2=-4\frac{h_1}{h_3}$, and $g_3=-4\frac{h_0}{h_3}$.

The corresponding nonlinearity and the potential in (4)–(6) has to be given by the form

$$g(x,t)=\frac{5h_1^4\gamma(1+\lambda\,\mathrm{e}^{-\gamma^2x^2})^3}{128h_0^3C_2^2}, \quad (29)$$

$$v(x,t)=w(t)x^2-\frac{\lambda\gamma^4(2\mathrm{e}^{\gamma^2x^2}-\lambda)x^2}{(\mathrm{e}^{\gamma^2x^2}+\lambda)^2}-\frac{3h_1^2\gamma^2(\mathrm{e}^{\gamma^2x^2}+\lambda)^2}{8h_0\,\mathrm{e}^{2\gamma^2x^2}}+\frac{\lambda\gamma^2}{\mathrm{e}^{\gamma^2x^2}+\lambda}-\rho_t. \quad (30)$$

In Figure 4, we still plot $w(t)$ in two cases to illustrate its periodic and quasiperiodic features. In Figure 5, we

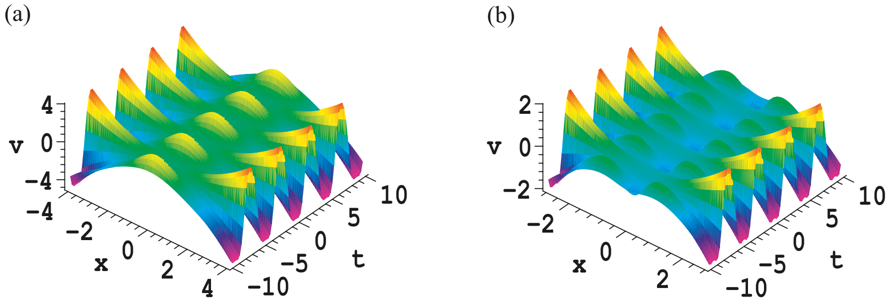

Fig. 2 (colour online). Plots of $v(x,t)$ in (27). (a) $\rho = 0$, $\lambda = 0.5$, (b) $\rho = 0$, $\lambda = -0.5$, and the other parameters are the same as the parameters in Figure 1: the line with the smaller amplitude for (a), the line with the larger amplitude for (b), respectively.

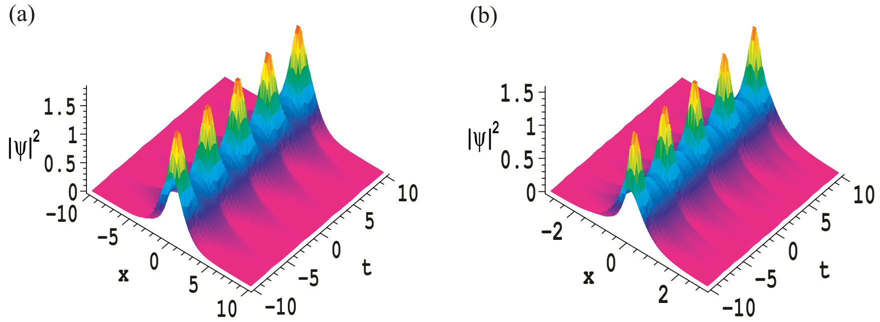

Fig. 3 (colour online). Plots of $|\psi|^2$ of solution (25), with $\lambda = \pm 0.5$, $m = 1$, $C_1 = 1$, $h_2 = 1$, $h_4 = -1$, and the other parameters are the same as the parameters in Figure 1: the line with the smaller amplitude for periodic (a), the line with the larger amplitude for quasiperiodic (b), respectively.

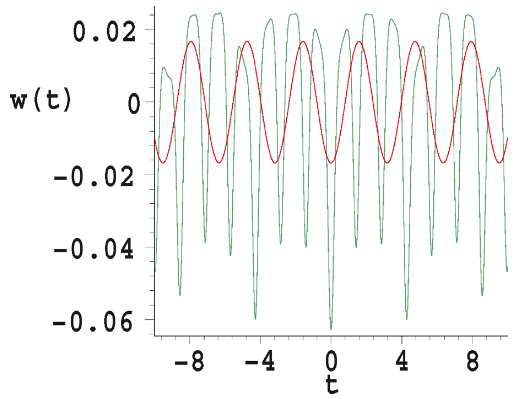

Fig. 4 (colour online). Plots of $w(t)$ in (6), with $\alpha_0 = 0.1$, $\alpha = 1$, $\beta = 0$, $k_1 = 1$, $n_1 = 0.2$ for the line with smaller amplitude, and with $\alpha_0 = 0.1$, $\alpha = 0.03$, $\beta = 0.01$, $k_1 = 1$, $k_2 = 3.2$, $n_1 = n_2 = 0.9$ for the line with larger amplitude, respectively.

note that the potentials periodically oscillate from attractive to expulsive behaviour mainly. But here the attractive or expulsive structures near the origin are obvious and do not change if $w(t)$ ist periodically or quasiperiodically. In this case, in Figures 6 and 7, if we change $\lambda \to -\lambda$, the periodic and quasiperiodic dark solitons become bright solitons, but at the same time maintain the same shape at the condition of the same potential.

3.2. Moving Solitons

In fact, to observe the moving solitons, we must present solutions of (1) when the center of mass of the soliton moves with non-zero velocity. In order to arrive

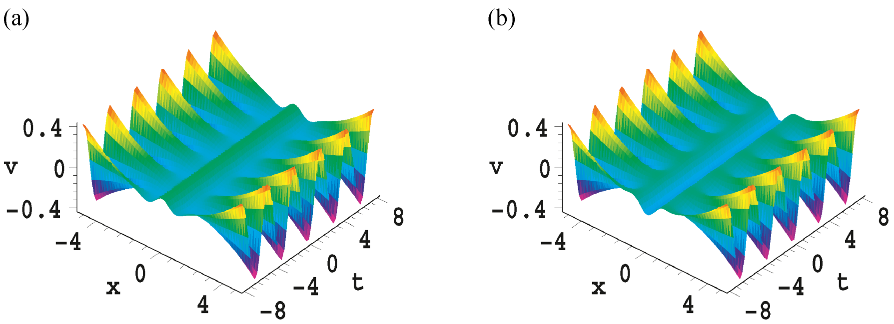

Fig. 5 (colour online). Plots of $v(x,t)$ in (30). (a) $\rho = 0$, $\lambda = 0.05$ and $w(t)$ is the same as in Figure 4 (small amplitude); (b) $\rho = 0$, $\lambda = -0.05$, and $w(t)$ is the same as in Figure 4 (large amplitude).

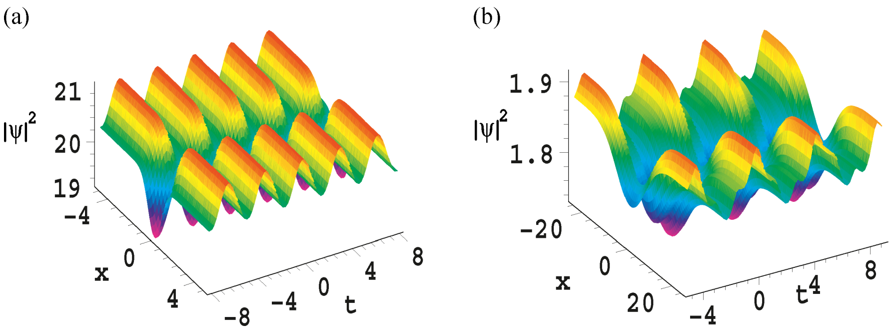

Fig. 6 (colour online). Plots of $|\psi|^2$ of solution (28) with $\lambda = 0.05$, $C_2 = 0.01$, $h_0 = 0.1$, $h_1 = 0.001$, the other parameters are the same as in Figure 4, the small amplitude line for periodic (a) and the large amplitude line for quasiperiodic (b), respectively.

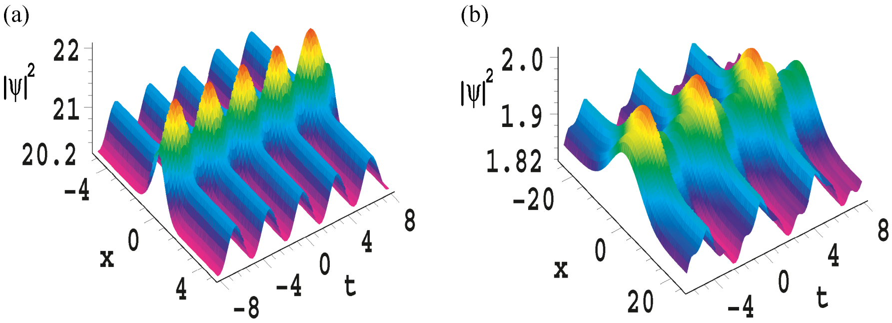

Fig. 7 (colour online). Plots of $|\psi|^2$ of solution (28) with $\lambda = -0.05$, $C_2 = 0.01$, $h_0 = 0.1$, $h_1 = 0.001$, the other parameters are the same as periodic (a) Figure 4a (small amplitude) and quasiperiodic (b); Figure 4b (large amplitude), respectively.

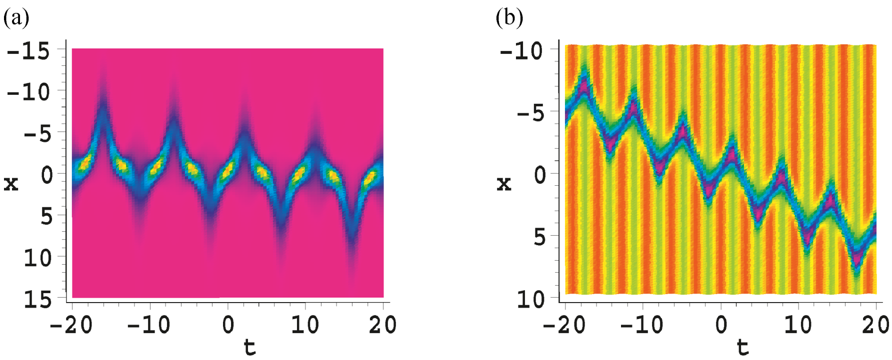

Fig. 8 (colour online). (a) Density plot of $|\psi|^2$ of solution (25) with the parameters $m = 1, C_1 = \sigma_0 = k_1 = h_2 = -h_4 = \alpha = 1$, $\alpha_0 = 0, n_1 = 0.9$, $\lambda = 0.1$; (b) Density plot of $|\psi|^2$ of solution (28) with the parameters $C_2 = 0.01$, $\sigma_0 = k_1 = \alpha = 1$, $h_0 = 0.1$, $h_1 = -0.001$, $\alpha_0 = 0$, $n_1 = 0.2, \lambda = 0.1$.

this aim, we set $f(t) = 0$ and $\sigma(t) \neq 0$ in (5) and (6), then we derive

$$\sigma(t) = \sigma_0 \int \gamma(t)^2 \, \mathrm{d}t, \tag{31}$$

where σ_0 is an arbitrary constant.

Obviously, the center of mass of the solition will move in a complex way according to (31). In Figure 8, we show the moving track of (25) and (28) while the center of mass of the solition moves according to (31).

4. Summary and Discussion

In this paper, by extending the generalized sub-equation method, we present three families of analytical solutions of the one-dimensional nonlinear Schrödinger equation with potentials and nonlinearities depending on time and on spatial coordinates. Then, based on these solutions, periodically and quasiperiodically oscillating solitons (dark and bright) and moving solitons are observed. At the same time, at different choice of potentials and nonlinearities, features of soltion solutions are discussed. These results provide some potential applications in many physical fields, such as Bose–Einstein condensate, nonlinear optics, etc., and open up opportunities for further studies on relative experiments, such as controlling Bose–Einstein condensates by designing potentials and nonlinearities depending on time and space.

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